Some algebra in algebraic topology

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Overview of the presentation

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- What is this thing called algebraic topology

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- Developing a project proposal for a research grant

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Circle examples:

- $1 : S^1 \longrightarrow S^1,\ e^{i\theta} \mapsto e^{i\theta}$ for each $0 \leq \theta \leq 2\pi$ in the complex plane.
- $2 : S^1 \longrightarrow S^1,\ e^{i\theta} \mapsto e^{2i\theta}$
- $0 : S^1 \longrightarrow S^1,\ e^{i\theta} \mapsto e^{0}$
- $-1 : S^1 \longrightarrow S^1,\ e^{i\theta} \mapsto e^{-i\theta}$

Homotopy

The set of all maps from a circle to itself $[S^1, S^1] := \pi_1(S^1) \cong \mathbb{Z}$ with the (abelian) group addition being the composition of number of times the circle is wrapped around itself.

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Any map between spaces $f : X \to Y$ induces a homomorphism $f_* : \pi_n(X) \to \pi_n(Y)$ for each $n \in \mathbb{N}$. Homotopy is a covariant functor.

Cohomology

For any group G we can define a family of Eilenberg-Mac Lane spaces $K(G,n)$ with the property that $\pi_m\left(K(G,n)\right) = \begin{cases} G, & m = n \\ 0, & m \neq n \end{cases}$

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They can also be defined combinatorially for a triangulable space via the homology $H^n(X) =$ Ker d_n / Im d_{n-1} of the cochain complex

$$\longrightarrow C^{n-1}(X) \xrightarrow{d_{n-1}} C^n(X) \xrightarrow{d_n} C^{n+1}(X) \longrightarrow$$

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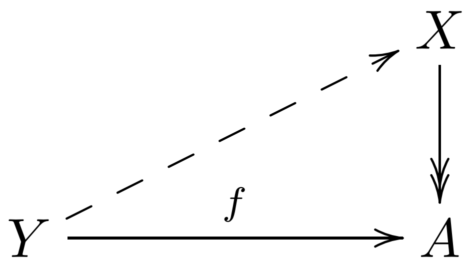
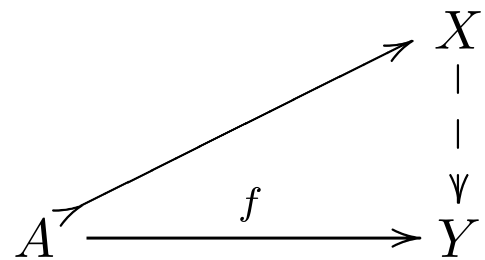
Unlike homotopy and homology, any map between spaces $f : X \to Y$ induces a homomorphism $f^* : H^n(Y) \to H^n(X)$ for each $n \in \mathbb{N}$. Cohomology is a contravariant functor.

What's the point?

Many geometrical problems can be formulated as lifting or extension problems.





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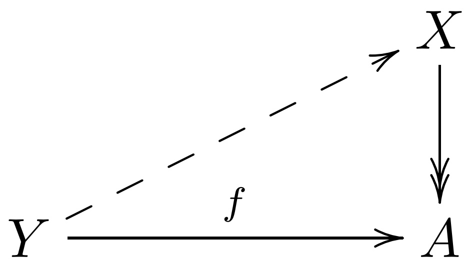
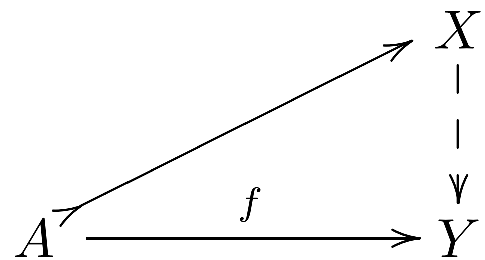
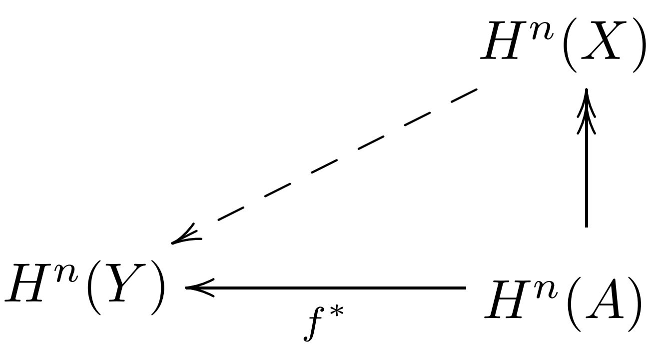
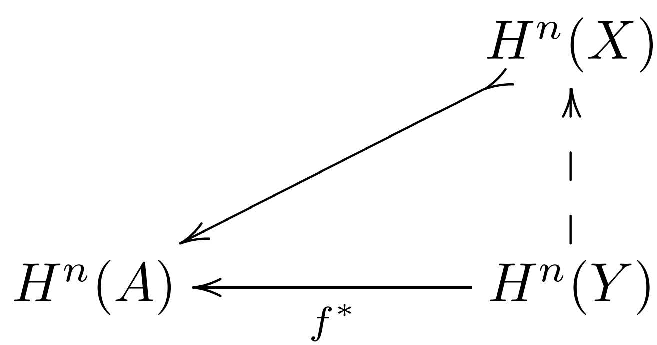




These can now be reformulated as a more tractible algebraic problem





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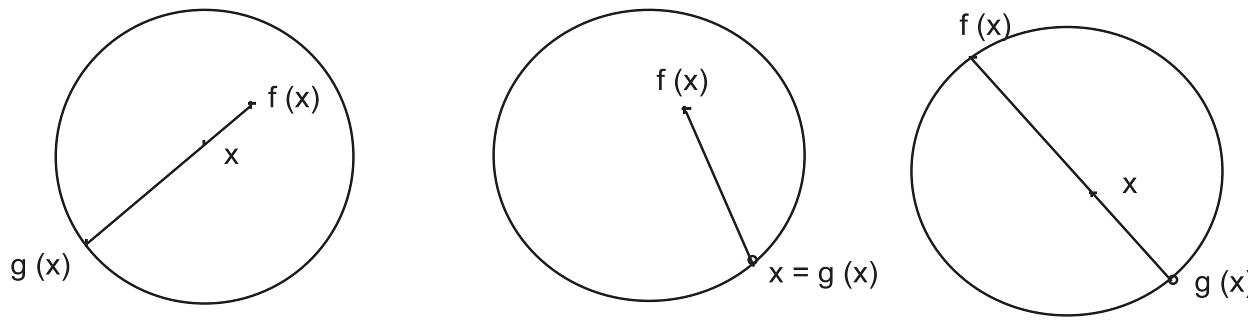

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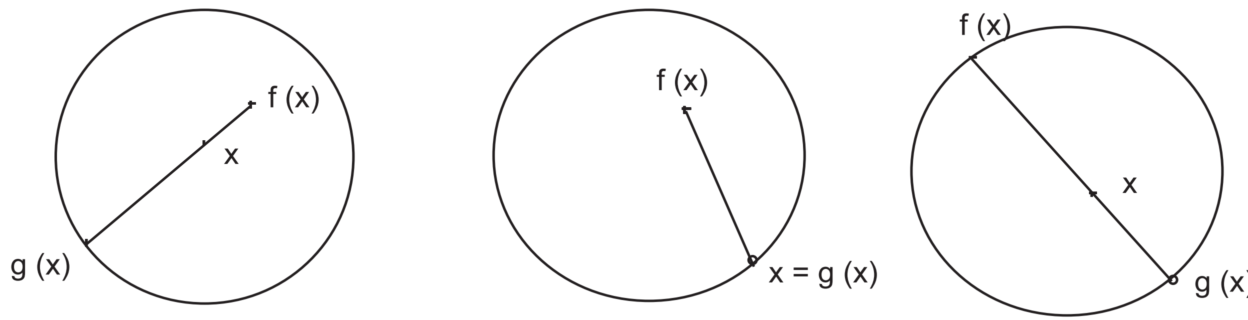

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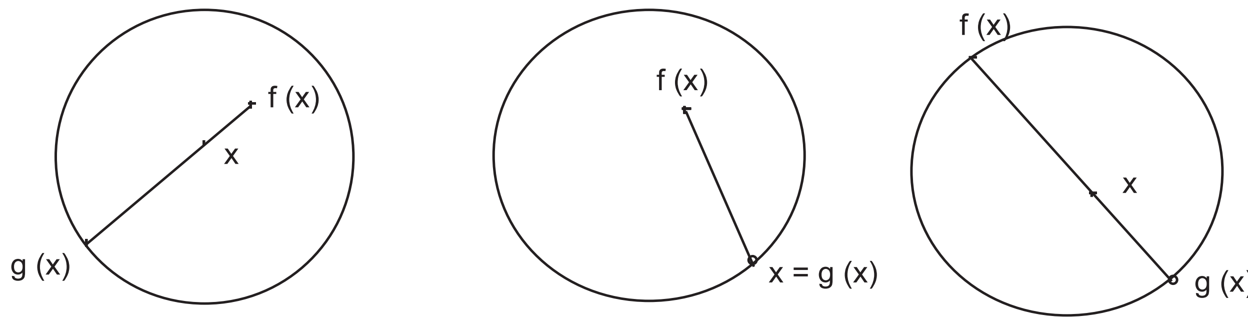

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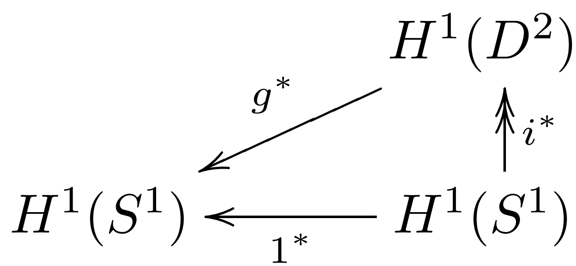
Then $g \circ i = 1_{S^1}$ and we have

$$\begin{array}{ccc} & & D^2 \\ & \overset{i}{\nearrow} & \downarrow g \\ S^1 & \xrightarrow{1} & S^1 \end{array}$$

Taking the ($n = 1$) cohomology of this diagram gives



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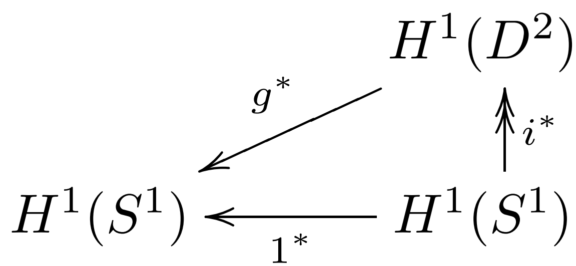

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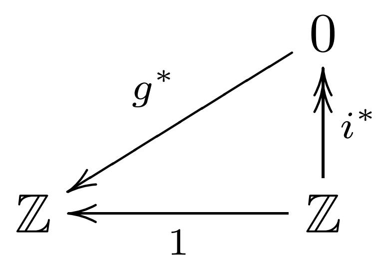

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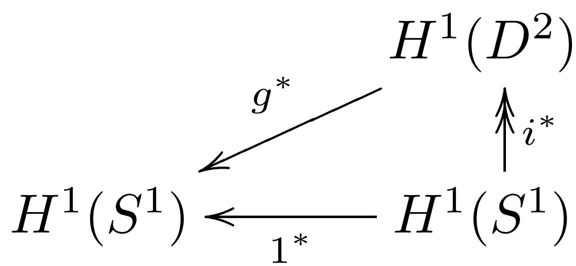

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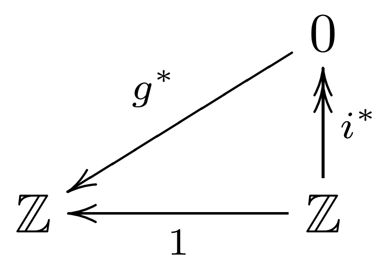

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$$\begin{array}{ccc} & & H^1(D^2) \\ & \overset{g^*}{\swarrow} & \uparrow i^* \\ H^1(S^1) & \underset{1^*}{\longleftarrow} & H^1(S^1) \end{array}$$

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Hence the assumption that, for every x in the disc, $x \neq f(x)$ cannot be true, so any $f : D^2 \to D^2$ must have a fixed point.

Consequences of Brouwer's theorem

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- you can't comb a tennis ball without a part or crown (can comb a circle or torus)
- at any point in time there is a cyclone somewhere in the world (though the strength is not determined)
- there is always a point on the sun's surface from which no light is emitted (doesn't shine everywhere)

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The more we know about the structure of $H^*(X)$, the more tractible the algebraic problems become hence we're more able to solve geometrical problems.

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Relations on the integral operations have eluded explicit formulation except for (can be given implicitly as a functor)

$$\begin{aligned} (x+y)\circ z &= (x\circ z)+(y\circ z) \\ \text{and} \quad (x\cup y)\circ z &= (x\circ z)\cup(y\circ z) \end{aligned}$$

and bilinearity and anticommutivity of cup product

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If any of the above are acheived: To what problems could this new knowledge be applied?

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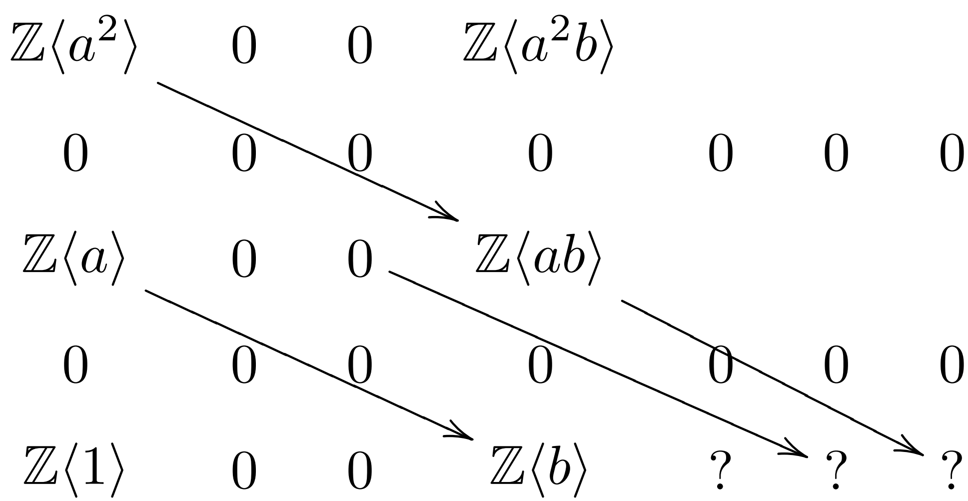
For Eilenberg-Mac lane spaces this spectral sequence is
$E_2^{p,q} = H^q(\mathbb{Z}, n-1) \otimes H^p(\mathbb{Z}, n) \Longrightarrow 0,\ p, q \neq 0$ and $E_\infty^{0,0} = \mathbb{Z}$

As much fun as you can have

The third page of the spectral sequence for $K(\mathbb{Z}, 3)$ is identical to the second and starts off



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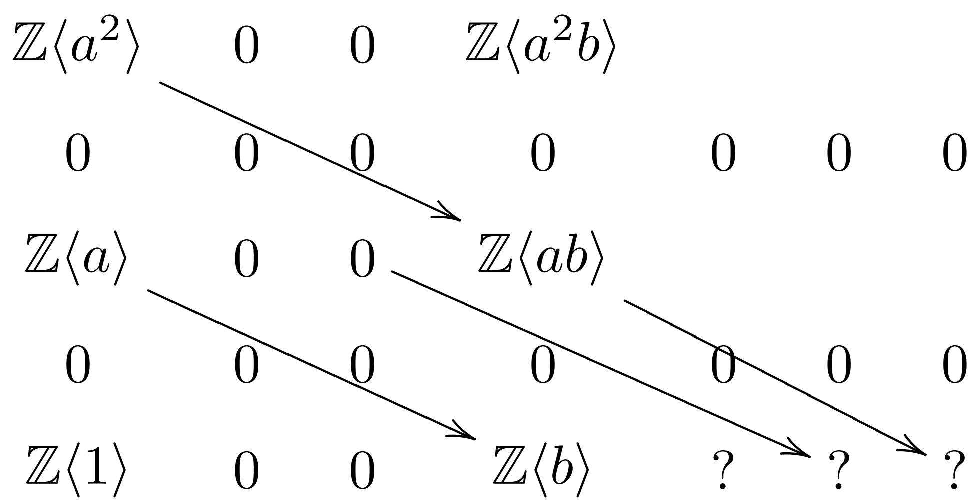

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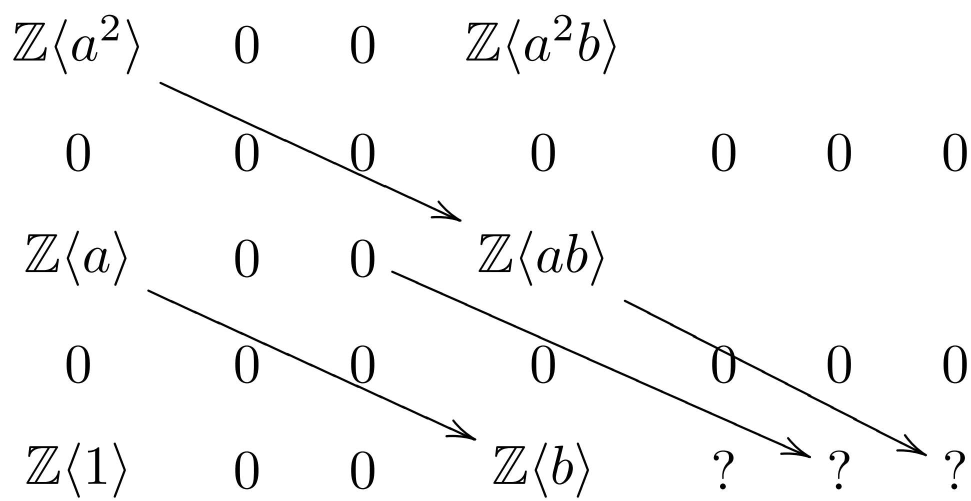

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The cup product structure is implicit in the spectral sequence but many techniques (incuding the known relations, stability and later calculated groups) are necessary to determine the other generators

Table of $H^m(\mathbb{Z}, n)$

m						
13	0	$\mathbb{Z}/2\langle bd\rangle$	$\mathbb{Z}/5\langle j\rangle$ $\oplus\ \mathbb{Z}/3\langle fh\rangle$	$\mathbb{Z}/2\langle k\Omega^{-1}g\rangle$	$\mathbb{Z}/2\langle q\rangle$	0
12	$\mathbb{Z}\langle a^6\rangle$	$\mathbb{Z}/2\langle b^4\rangle \oplus \mathbb{Z}/5\langle e\rangle$	$\mathbb{Z}\langle f^3\rangle$	$\mathbb{Z}/2\langle m\rangle$	$\mathbb{Z}\langle n^2\rangle$	
11	0	$\mathbb{Z}/3\langle bc\rangle$	$\mathbb{Z}/2\langle fg\rangle$ $\oplus\ \mathbb{Z}/2\langle ?\rangle$	0	$\mathbb{Z}/2\langle o\rangle$ $\oplus\ \mathbb{Z}/3\langle p\rangle$	
10	$\mathbb{Z}\langle a^5\rangle$	$\mathbb{Z}/2\langle d\rangle$	0	$\mathbb{Z}/2\langle k^2\rangle \oplus \mathbb{Z}/3\langle l\rangle$		
9	0	$\mathbb{Z}/2\langle b^3\rangle$	$\mathbb{Z}/3\langle h\rangle$	0		
8	$\mathbb{Z}\langle a^4\rangle$	$\mathbb{Z}/3\langle c\rangle$	$\mathbb{Z}\langle f^2\rangle$			
7	0	0	$\mathbb{Z}/2\langle g\rangle$			$\langle r\rangle$
6	$\mathbb{Z}\langle a^3\rangle$	$\mathbb{Z}/2\langle b^2\rangle$			$\langle n\rangle$	
5	0	0		$\langle k\rangle$		
4	$\mathbb{Z}\langle a^2\rangle$		$\langle f\rangle$			
3	0	$\langle b\rangle$				
2	$\mathbb{Z}\langle a\rangle$					
	2	3	4	5	6	7

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In contrast to the cohomology algebras over the finite fields

- The product $b \cup b \cup c$ in the group $H^{14}(\mathbb{Z}, 4)$ is trivial (relation)
  Question: Why does Cartan's 'method of constructions' fail to work for integral cohomology?
- There is more than one stable operation (of differing order) given in $H^{11}(\mathbb{Z}, 6)$

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There are many sub-studies amongst this general framework

- linearity, commutativity of cross-cap products - Honours level
- extending the table of cohomology groups of Eilenberg-MacLane spaces - Masters level
- Formulating relations or significant progress towards their formulation - PhD level