EXAMPLES OF Π-ALGEBRAS

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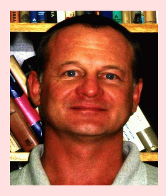
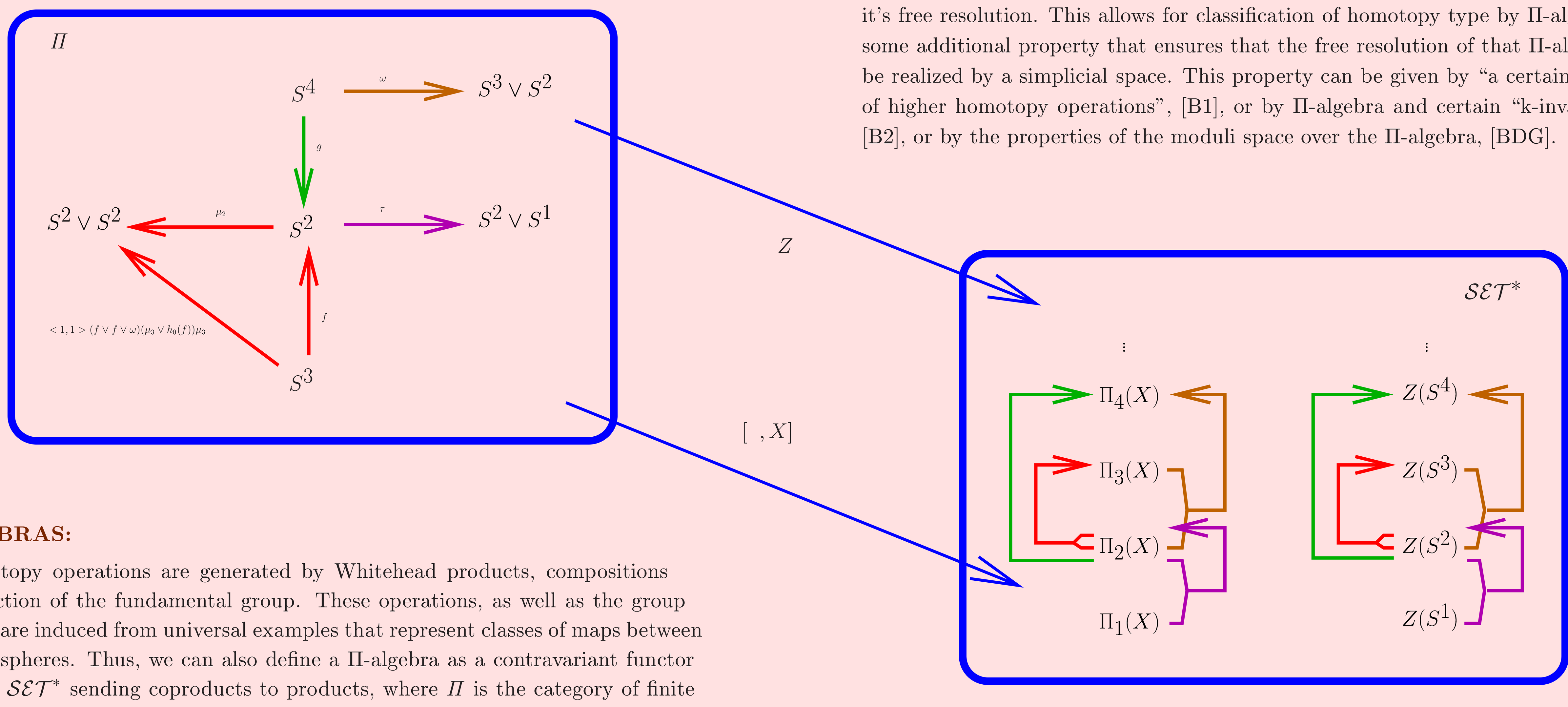


ABSTRACT: The homotopy Π-algebra of a pointed topological space, X, consists of the homotopy groups of X together with the additional structure of the primary homotopy operations. We extend two well known results for homotopy groups to homotopy Π-algebras and look at some examples illustrating the depth of structure on homotopy groups; from graded group to graded Lie ring, to Π-algebra and beyond. We also describe an abstract Π-algebra and give three abstract Π-algebra structures on the homotopy groups of the loop space of X which can be realized as the homotopy Π-algebras of three different spaces.



INTRODUCTION:

The homotopy groups of a space, X, together with the primary homotopy operations on them is called the homotopy Π-algebra of X and is denoted $\Pi_*(X)$. For each X, we can construct a simplicial space that realizes a free simplicial resolution of $\Pi_*(X)$. We can describe an abstract Π-algebra as a collection of groups with operations satisfying the properties of homotopy operations. An abstract Π-algebra can be realized as the homotopy Π-algebra of a space if there is a simplicial space realizing it's free resolution. This allows for classification of homotopy type by Π-algebra and some additional property that ensures that the free resolution of that Π-algebra can be realized by a simplicial space. This property can be given by "a certain sequence of higher homotopy operations", [B1], or by Π-algebra and certain "k-invariants", [B2], or by the properties of the moduli space over the Π-algebra, [BDG].

Π-ALGEBRAS:

The homotopy operations are generated by Whitehead products, compositions and the action of the fundamental group. These operations, as well as the group additions, are induced from universal examples that represent classes of maps between wedges of spheres. Thus, we can also define a Π-algebra as a contravariant functor $Z : \mathit{\Pi} \longrightarrow \mathcal{SET}^*$ sending coproducts to products, where $\mathit{\Pi}$ is the category of finite wedges of spheres and classes of maps between them. In the particular case of the homotopy Π-algebra of X, this functor is $[\ \ , X]$, assigning to each object of $\mathit{\Pi}$ the set of homotopy classes of maps from it into X.

In the diagram above, the universal examples for Whitehead product and the action of the fundamental group are denoted ω and τ. The universal example for a composition is any representing map of an element in $\Pi_m(S^n)$. The comultiplication on the n-sphere is denoted μ_n. Observe that compatability of operations is implicit in the category $\mathit{\Pi}$. Specifically, $\mu_2 f = <1,1> (f \vee f \vee \omega)(\mu_3 \vee h_0(f))\mu_3$ is a statement of Hilton's formula $(x+y)f = xf + yf + [x,y]h_0(f)$.

REFERENCES:

[B1] David Blanc. Higher homotopy operations and the realization of homotopy groups. *Proc. London Math. Soc.*, 3(70): 214 - 240, 1995.

[B2] David Blanc. Algebraic invariants for homotopy types. *Math. Proc. Camb. Philos. Soc.*, 127: 497 - 523, 1999.

[BDG] David Blanc, Bill Dwyer and Paul Goerss. The realization space of a Π-algebra. *Preprint*, 2001.

[T] Hiroshi Toda. *Composition methods in homotopy groups of spheres.* Princeton University Press, New Jersey, 1962.

EXAMPLES:

By taking the n^{th} Postnikov section of a simply connected space we can truncate the Π-algebra at dimension n and dismiss the action of the fundamental group. Defining the product of Π-algebras as the direct sum of their underlying abelian groups with operations acting componentwise, we show that $\Pi_*(X \times Y) \cong \Pi_*(X) \sqcap \Pi_*(Y)$ and hence we can construct generalised Eilenberg-MacLane spaces with any desired homotopy groups and a trivial primary operation structure. This is called a trivial Π-algebra. With reference to the classification problem, we note that $(\mathbb{C}P^2)^{[5]}$ has the same trivial Π-algebra as a generalised Eilenberg-MacLane space. However, these spaces are not homotopy equivalent since there is a non-trivial secondary Whitehead product operation on the homotopy groups of $(\mathbb{C}P^2)^{[5]}$ which is trivial on the homotopy groups of the generalised Eilenberg-MacLane space.

In the examples below, K is the generalised Eilenberg-MacLane space giving the trivial Π-algebra on the homotopy groups of a space X, and $\eta_i = \Sigma^{i-2}\eta_2$ is the suspended Hopf map. We display only the generating compositions, [T].

ABSTRACT Π-ALGEBRAS:

For any operation $\theta : \Pi_n(X) \times \Pi_m(X) \longrightarrow \Pi_p(X)$ we can define an operation $\tilde{\theta} : \Pi_{n-1}(\Omega X) \times \Pi_{m-1}(\Omega X) \longrightarrow \Pi_{p-1}(\Omega X)$ using the adjunction of Ω and Σ. Thus we can induce a Π-algebra structure on the homotopy groups of ΩX that is realized by the space X. This Π-algebra is denoted $\Pi_{*-1}(\Omega X)$. It's Lie ring structure is given by the Samelson product. On the other hand, the homotopy Π-algebra, $\Pi_*(\Omega X)$, will have no Lie ring structure since Ω annihilates Whitehead product and composition operations will be given by stable maps only. This is because $\widetilde{\Sigma f} = f$ and hence in the diagram below $\widetilde{\eta_{i+1}} = \eta_i$. We can also define all operations to be trivial and give the trivial Π-algebra on the homotopy groups of ΩX which will be realized by a generalised Eilenberg-MacLane space.