Integral cohomology operations and relations

Andrew Percy

`andrew.percy@sci.monash.edu.au`

School of Applied Science and Engineering

Monash University

Gippsland Campus

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There is a richer algebraic structure on the graded cohomology groups of a topological space, $\tilde{H}^*(X)$, in the form of cohomology operations and relations between these operations. Cup product is a well known example of a cohomology operation. $\tilde{H}^*(X)$ is known to be a graded ring.

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As a general rule: More structure = more sensitivity for calculations. For instance, a map $f : X \to Y$ must induce a graded homomorphism $f^* : H^*(Y) \to H^*(X)$ but if we include cup product with the graded cohomology groups then f^* must be a graded ring homomorphism. See Steenrod (1972)

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The case for cohomology over $\mathbb{Z}/2$ coefficients is well known. The stable operations and relations form a Hopf algebra, called the Steenrod algebra, and the graded group $H^*(X; \mathbb{Z}/2)$ forms an algebra over the Steenrod algebra. These $\mathbb{Z}/2$ cohomology algebras have been effectively used: Serre (1951), Adams (1960).

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For notational simplicity we write K^n for $K(\mathbb{Z}, n)$ and $\tilde{H}^n(K^p)$ for $\tilde{H}^n(\mathbb{Z}, p)$. Then $\tilde{H}^n(X)$ can be defined as the homotopy classes of maps $[X, K^n]$.

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Primary cohomology operations are in bijection with elements of the cohomology groups of finite products of Eilenberg-Mac Lane spaces. Thus, for any operation $\theta : \tilde{H}^p(X) \times \tilde{H}^q(X) \to \tilde{H}^m(X)$ there is a map $h_\theta \in \tilde{H}^m(K^p \times K^q) = [K^p \times K^q, K^m]$ which gives the operation by composition. The example of cup product follows.

Cup product as a map

The cup product is in bijective correspondence with an element $h_\cup \in \tilde{H}^{p+q}(K^p \times K^q)$ which can be considered a map $h_\cup : K^p \times K^q \to K^{p+q}$. Then given $x \in \tilde{H}^p(X)$ and $y \in \tilde{H}^q(X)$ the cup product $x \cup y \in \tilde{H}^{p+q}(X)$ is given by $h_\cup \circ \{x, y\}$ as in the diagram:

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$$\begin{tikzcd}
 & K^p & \\
X \arrow[ur, "x"] \arrow[r, "{\{x,y\}}"] \arrow[dr, "y"'] & K^p \times K^q \arrow[u] \arrow[d] \arrow[r, "h_\cup"] & K^{p+q} \\
 & K^q &
\end{tikzcd}$$

Co-Π-algebras as functors

The object of graded cohomology group $\tilde{H}^*(X)$ together with all primary operations and the relations between them is Eckmann-Hilton dual to the Π-algebras of homotopy theory. They can therefore be encoded in a functor $[X,\ ]$ from the category of finite products of Eilenberg-Mac Lane spaces to the category of pointed sets.

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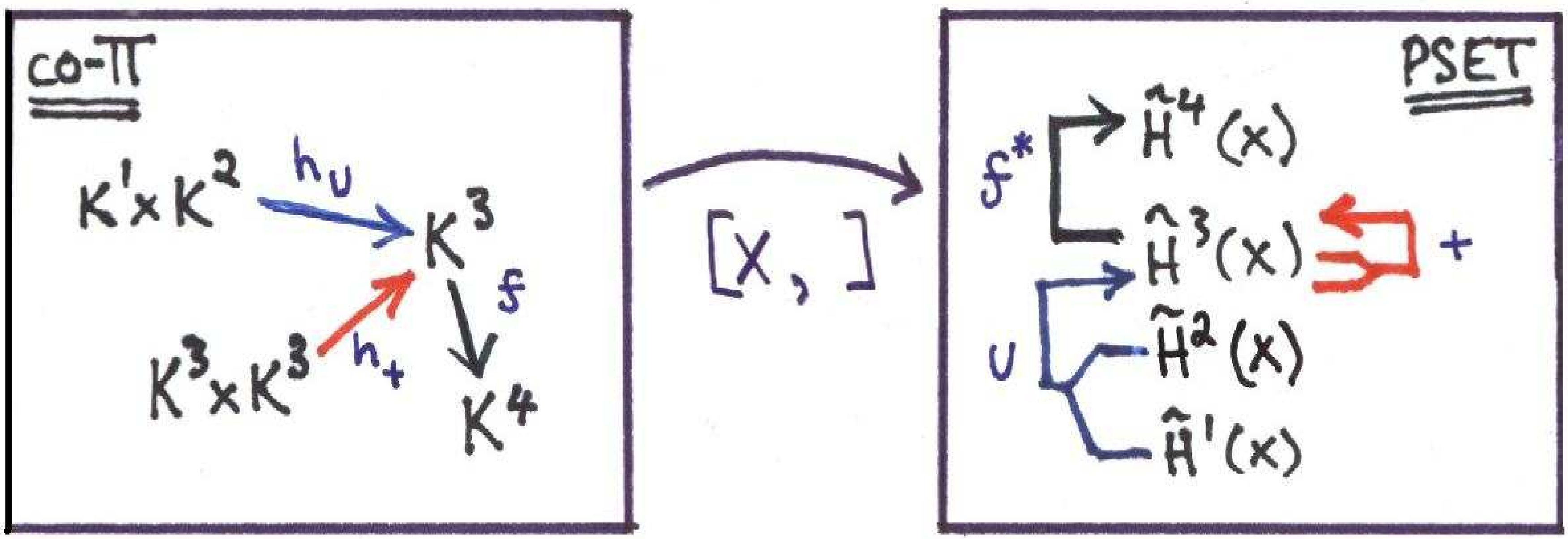

Where to find relations

The functor description is particularly useful to show that $\tilde{H}^*(K^n)$ is a free co-Π-algebra (in fact finite products of EM spaces). See Percy (2004). $[\mathcal{F}G^*, \tilde{H}^*(X)] \cong [G^*, \mathcal{U}\tilde{H}^*(X)]$ for G^* a graded abelian group. This means that all operations acting on any elements in G^* will give non-trivial cohomology classes in $\mathcal{F}G^*$, unless a relation somehow prevents this.

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Since $\tilde{H}^*(K^n)$ is a free co-Π-algebra these groups provide a place to look for examples of relations.

Generators of integral operations

The Künneth theorem together with a splitting of the long exact cohomology sequence yields

$$\begin{aligned}\tilde{H}^n(K^p \times K^q) \quad &\cong \quad \tilde{H}^n(K^p) \oplus \tilde{H}^n(K^q) \\ &\bigoplus_{i+j=n} \tilde{H}^i(K^p) \otimes \tilde{H}^j(K^q) \\ &\bigoplus_{i+j=n+1} \operatorname{Tor}\left(\tilde{H}^i(K^p), \tilde{H}^j(K^q)\right)\end{aligned}$$

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From this decomposition we show that all operations are generated by compositions (single groups), cup products (the tensor terms) and cross-cap products (the Tor terms). See Percy (2004) or Eilenberg and Mac Lane (1954).

Cross-cap products

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For $x \in H^i(K^p) \cong \mathbb{Z}/s$ and $y \in H^j(K^q) \cong \mathbb{Z}/t$ we get an element $x \bullet y \in H^{i+j-1}(K^p \times K^q)$ given in the Tor summand $\mathbb{Z}/\mathrm{gcd}(s,t)$ of the Künneth formula.

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If a cocycle $x \in C^i(K^p)$ has s-torsion then there is a coboundary $\alpha \in C^{i-1}(K^p)$ with $\delta(\alpha) = sx$. Similarly, there is $\beta \in C^{j-1}(K^q)$ with $\delta(\beta) = ty$. If $\text{gcd}(s,t) = l$ with $gl = s$ and $hl = t$, then there is a cocycle $x \bullet y = gx \times \beta + (-1)^i \alpha \times hy$ where, on the right, $\times$ is the cochain cross product which induces the cohomology cross product $\bar{\times}$ also known as the external cup product.

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Since α and β need not be cocycles describing cross-cap operations as a diagram of maps may not be possible.

Known relations

If we have operations from suitable groups, so that the equations below are well defined, then some known relations are

$$
\begin{aligned}
(a+b)\circ z &= (x\circ z)+(y\circ z) && (1)\\
\text{and}\quad (a\cup b)\circ z &= (a\circ z)\cup(b\circ z) && (2)\\
\text{for } z \text{ primitive (including stable)}\quad z\circ(a+b) &= (z\circ a)+(z\circ b) && (3)\\
a\cup z = -(z\cup a)\quad \text{and}\quad z\cup(a+b) &= (z\cup a)+(z\cup b) && (4)
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The method of calculating the groups $\tilde{H}^m(K^n)$ is chosen for simplicity and because we gain significant information about generators for many of the groups.

The Leray-Serre spectral sequence

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The spectral sequence simplifies (within the regions used) to

$$E_2^{p,q} = \tilde{H}^q(K^{n-1}) \otimes \tilde{H}^p(K^n) \Longrightarrow 0$$

with the cup product of the generators of $\tilde{H}^q(K^{n-1})$ and $\tilde{H}^p(K^n)$ identified with the spectral sequence multiplication so that we write ab for $a \cup b$ and a^2 for $a \cup a$.

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We can start with $S^1 \to PK^2 \to K^2$ to calculate $\tilde{H}^*(K^2)$ and then use $K^2 \to PK^3 \to K^3$ and so on.

The groups $H^m(K^n)$

m						
13	$\mathbb{Z}/2\langle a?\rangle$	$\mathbb{Z}/5\langle ?\rangle \oplus ?$			0	
12	$\mathbb{Z}/2\langle a^4\rangle \oplus \mathbb{Z}/5\langle ?\rangle$	$\mathbb{Z}\langle e^3\rangle \oplus ?$		$\mathbb{Z}\langle k^2\rangle$		
11	$\mathbb{Z}/3\langle a?\rangle$	$\mathbb{Z}/2\langle e?\rangle$ $\oplus \mathbb{Z}/2\langle ?\rangle$	0	$\mathbb{Z}/2\langle ?\rangle$ $\oplus \mathbb{Z}/3\langle ?\rangle$		
10	$\mathbb{Z}/2\langle ?\rangle$	0	$\mathbb{Z}/2\langle i^2\rangle \oplus \mathbb{Z}/3\langle ?\rangle$			
9	$\mathbb{Z}/2\langle a^3\rangle$	$\mathbb{Z}/3\langle ?\rangle$	0			
8	$\mathbb{Z}/3\langle ?\rangle$	$\mathbb{Z}\langle e^2\rangle$				
7	0	$\mathbb{Z}/2\langle ?\rangle$			$\langle n\rangle$	
6	$\mathbb{Z}/2\langle a^2\rangle$			$\langle k\rangle$		
5	0		$\langle i\rangle$			
4	0	$\langle e\rangle$				
3	$\langle a\rangle$					
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Determining the generators

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Similarly we can determine that $\tilde{H}^{10}(K^3)$, $\tilde{H}^7(K^4)$, $\tilde{H}^9(K^4)$ and the undetermined generators in the summands of $\tilde{H}^{10}(\mathbb{Z}, 5)$ and $\tilde{H}^{11}(\mathbb{Z}, 6)$ are indecomosable since all other possibilities involve operations from trivial groups.

Using known relations

The undetermined generator for the $\mathbb{Z}/5$ summand in $\tilde{H}^{12}(K^3)$ could be given by an indecomposable operation or either $x \circ a^2$ with $x \in \tilde{H}^{12}(K^6)$ or $y \circ a^3$ with $y \in \tilde{H}^{12}(K^9) \cong \mathbb{Z}/2$.

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Since y has order 2 the relation in equation (1) gives us $0 = 0 \circ b^3 = (t+t) \circ b^3 = (t \circ b^3) + (t \circ b^3) = 2(t \circ b^3)$ so that $y \circ a^3$ could not generate a summand of $\mathbb{Z}/5$. However, $t \circ b^3$ could be related to a^4.

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With $x \in \tilde{H}^{12}(K^6) \cong \mathbb{Z}\langle k^2 \rangle$ we have $x = \alpha k^2$ for $\alpha \in \mathbb{Z}$. Then using relations (1), (2) and the fact that $k : K^6 \to K^6$ is the identity map we get $\left(\alpha k^2\right) \circ a^2 = \alpha \left(k \cup k\right) \circ a^2 = \alpha \left(\left(k \circ a^2\right) \cup \left(k \circ a^2\right)\right) = \alpha a^4$ so that $x \circ a^2$ is not in the $\mathbb{Z}/5$ summand.

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$\tilde{H}^{13}(K^4)$ has a summand $\mathbb{Z}/5\langle ? \rangle$. The only possible decomposable generator is a cross-cap product $(f \bullet f) \circ \{e, e\}$. However, since gcf$(2,2) = 2$, $f \bullet f$ has order 2 and hence $0 = 0 \circ \{e, e\} = (f \bullet f + f \bullet f) \circ \{e, e\} = 2\left((f \bullet f) \circ \{e, e\}\right)$.Thus, $(f \bullet f) \circ \{e, e\}$ has order 2 and hence cannot generate this summand.

The “completed” table

m						
13	$\mathbb{Z}/2\langle ac\rangle$	$\mathbb{Z}/5\langle h\rangle \oplus ?$			0	
12	$\mathbb{Z}/2\langle a^4\rangle \oplus \mathbb{Z}/5\langle d\rangle$	$\mathbb{Z}\langle e^3\rangle \oplus ?$		$\mathbb{Z}\langle k^2\rangle$		
11	$\mathbb{Z}/3\langle ab\rangle$	$\mathbb{Z}/2\langle ef\rangle$ $\oplus \mathbb{Z}/2\langle ?\rangle$	0	$\mathbb{Z}/2\langle l\rangle$ $\oplus \mathbb{Z}/3\langle m\rangle$		
10	$\mathbb{Z}/2\langle c\rangle$	0	$\mathbb{Z}/2\langle i^2\rangle \oplus \mathbb{Z}/3\langle j\rangle$			
9	$\mathbb{Z}/2\langle a^3\rangle$	$\mathbb{Z}/3\langle g\rangle$	0			
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Examples of relations

The group $\tilde{H}^{11}(K^3)$ provides two interesting examples. We may expect the element $m \circ a^2$ to generate a summand so we ask what is the relation of this operation to ab, which expresses a left distributivity of composition over cup product.

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$\tilde{H}^{13}(K^3) \cong \mathbb{Z}/2$ is generated by ac but there should also be an element $x \circ b$, with $x \in \tilde{H}^{13}(K^8)$, in this group. However $\tilde{H}^{13}(K^8)$ is in the stable range so x is $\Omega^{-2}l$ where Ω is the loop functor. The fact that $\Omega^{-2}l$ is stable combined with relation (3) tells us that $0 = \Omega^{-2}l \circ 0 = \Omega^{-2}l \circ (b + b + b) = 3(\Omega^{-2}l \circ b)$ and hence $\Omega^{-2}l \circ b = 0$ so this element is trivially related to ac.

Examples of relations

The group $\tilde{H}^{11}(K^3)$ provides two interesting examples. We may expect the element $m \circ a^2$ to generate a summand so we ask what is the relation of this operation to ab, which expresses a left distributivity of composition over cup product.

We also see that the cross-cap product $(a^2 \bullet a^2) \circ \{a, a\}$ being of order 2 must be a trivial multiple of ab which has order 3.

$\tilde{H}^{13}(K^3) \cong \mathbb{Z}/2$ is generated by ac but there should also be an element $x \circ b$, with $x \in \tilde{H}^{13}(K^8)$, in this group. However $\tilde{H}^{13}(K^8)$ is in the stable range so x is $\Omega^{-2}l$ where Ω is the loop functor. The fact that $\Omega^{-2}l$ is stable combined with relation (3) tells us that $0 = \Omega^{-2}l \circ 0 = \Omega^{-2}l \circ (b + b + b) = 3(\Omega^{-2}l \circ b)$ and hence $\Omega^{-2}l \circ b = 0$ so this element is trivially related to ac.

The generator of the $\mathbb{Z}/2\langle ? \rangle$ of $\tilde{H}^{11}(K^4)$ remains undetermined. There is the possibility that this group is generated by $\Omega^{-4}f \circ e^2$ with $\Omega^{-4}f \in \tilde{H}^{11}(K^8)$. We need to determine if there is a relation between $\Omega^{-4}f \circ e^2$ and fe. This would be another example of left distributivity of composition over cup product.

Related reading

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